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THE PERIODS OF EICHLER INTEGRALS FOR KLEINIAN GROUPS

BY

HIROKI SATO

ABSTRACT. We shall give period relations and inequalities for Eichler integrals for Kleinian groups Γ which have simply connected components of of the region of discontinuity of Γ. These are a generalization of those for abelian integrals. By using the period inequality, we shall give an alternate proof of a result of Kra.

0. Introduction. Let Γ be a nonelementary finitely generated Kleinian group, and Δ_1 a simply connected component of the region of discontinuity Ω of Γ.

M. Eichler [4], L V. Ahlfors [2], L. Bers [3] and I. Kra [5], [6] have represented periods of Eichler integrals as polynomials of degree at most $2q-2$, $q \geq 2$ being an integer. By this method, however, period relations for Eichler integrals are very complicated even when Γ is a Fuchsian group of the first kind (Eichler [4]). On the other hand, G. Shimura [7] has regarded the periods as column number vectors of length $2q-1$. In his paper he gave a certain period relation for Fuchsian groups.

By using Shimura's idea, we shall give period relations and inequalities for Eichler integrals for Kleinian groups. These are a generalization of those for abelian integrals. The main results in this paper are Theorems 1 and 2.

We shall state some notations in §1 and some lemmas in §2. In §3 we shall prove Theorem 1 and in §4 we shall state the period relations and inequalities, and give an alternate proof for the Kra result [5].

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1. Notation. Throughout this paper Γ denotes a nonelementary finitely generated Kleinian group with a simply connected component Δ_1 of the region of discontinuity Ω of Γ. We denote by Λ the limit set, $\lambda(z)|dz|$ the Poincaré metric on Ω. Let $q \geq 1$ be an integer. Set $\Delta = \bigcup_{A \in \Gamma} A(\Delta_1)$. It is a well-known fact (cf. [1]) that Δ/Γ is a Riemann surface which is obtained from a compact Riemann surface, denoted by $\overline{\Delta/\Gamma}$, by deleting a finite number of points. It is

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also known that Δ is a (disconnected) covering surface of Δ/Γ which ramifies over only a finite number of points.

We denote by $\mathbf{R}^n$ and $\mathbf{C}^n$ n-dimensional vector spaces over $\mathbf{R}$ and $\mathbf{C}$, respectively, $n \geq 0$ being an integer. We regard an element in $\mathbf{R}^n$ ($\mathbf{C}^n$) as a matrix with n rows and 1 column. We consider an element of Γ as a matrix $A = \binom{a\ b}{c\ d}$ with $ad - bc = 1$. We denote by $GL(m, \mathbf{C})$ the group of $m \times m$ invertible matrices over $\mathbf{C}$. Let $\binom{u}{v}$ be a vector in $\mathbf{C}^2$. For each $n = 2q - 2$, we denote by $\binom{u}{v}^n$ the vector in $\mathbf{C}^{n+1}$ whose components are $u^n, u^{n-1}, \cdots, uv^{n-1}, v^n$, where $\binom{u}{v}^0 = 1$. For $A \in \Gamma$ we set $\binom{u_A}{v_A} = A\binom{u}{v}$ and define $M(A) \in GL(n+1, \mathbf{C})$ by

$$\begin{pmatrix} u_A \\ v_A \end{pmatrix}^n = M(A)\begin{pmatrix} u \\ v \end{pmatrix}^n.$$

The following is due to Ahlfors [1]. Let $\Delta/\Gamma = S - \{p\}$ where S is a Riemann surface and $p \in S$. If there is a punctured neighborhood $M(p)$ of p such that π is unramified over $M(p)$, then there exists a parabolic transformation $A \in \Gamma$ with fixed point $s \in \Lambda$, and there is a Möbius transformation B with the following properties: (1) $B(\infty) = s$ and $B^{-1}AB(z) = z + 1$, $z \in \mathbf{C}$, (2) $B^{-1}(\Delta_1)$ contains a half-plane $U_{B^{-1}AB} = \{z \in \mathbf{C} \mid \mathrm{Im}\, z > c\}$, for some $c > 0$, (3) two points z_1 and z_2 of $B(U_{B^{-1}AB})$ are equivalent under Γ if and only if $z_2 = A^m(z_1)$ for some integer m, and (4) the image of $B(U_{B^{-1}AB})$ under π is a deleted neighborhood of p homeomorphic to a punctured disk. We call $W_A = B\{z \in \mathbf{C} \mid 0 \leq \mathrm{Re}\, z < 1, \mathrm{Im}\, z > c\}$ a cusped region belonging to p.

A mapping $\chi: \Gamma \to \mathbf{C}^{2q-1}$ is called a cocycle if $\chi_{AB} = {}^tM(B)\chi_A + \chi_B$ for $A, B \in \Gamma$, where ${}^tM(B)$ is the transposed matrix of $M(B)$. A cocycle $\chi: \Gamma \to \mathbf{C}^{2q-1}$ is called a coboundary if there exists $V \in \mathbf{C}^{2q-1}$ such that $\chi_A = {}^tM(A)V - V$ for any $\chi_A \in \mathbf{C}^{2q-1}$, $A \in \Gamma$. Then the first cohomology group $H^1(\Gamma, \mathbf{C}^{2q-1}, M)$ is the space of cocycles factored by the space of coboundaries. A cohomology class $P \in H^1(\Gamma, \mathbf{C}^{2q-1}, M)$ is called Δ-parabolic if, for every parabolic transformation $B \in \Gamma$ corresponding to a puncture on Δ/Γ, there is a $V \in \mathbf{C}^{2q-1}$ such that $P_B = {}^tM(B)V - V$ for some (and hence every) cocycle that represents P. The space of Δ-parabolic cohomology class is denoted by $PH^1_\Delta(\Gamma, \mathbf{C}^{2q-1}, M)$.

For an $m \times n$ matrix $N = (a_{ij})$ $(i = 1, 2, \cdots, n;\ j = 1, 2, \cdots, m)$ matrices $\overline{N}$ and $\widetilde{N}$ are defined by $\overline{N} = (\overline{a}_{ij})$ and $\widetilde{N} = (a_{m-i+1\,n-j+1})$, respectively, where $\overline{a}_{ij}$ is the complex conjugate of a_{ij}.

A holomorphic function ϕ on Δ is called an automorphic form of weight $(-2q)$ on Δ, $q \geq 1$, if $\phi(Az)A'(z)^q = \phi(z)$ for all $A \in \Gamma$. For $q \geq 2$ an automorphic form ϕ of weight $(-2q)$ on Δ is called integrable if

$$\iint_{\Delta/\Gamma} \lambda(z)^{2-q}|\phi(z)|\, dx\, dy < \infty.$$

We denote the Banach space of integrable automorphic forms on Δ by $A_q(\Delta, \Gamma)$. The form ϕ is called bounded if

$$\sup\{\lambda(z)^{-q}|\phi(z)|\ |z \in \Delta\} < \infty.$$

The Banach space of bounded automorphic forms on Δ is denoted by $B_q(\Delta, \Gamma)$. For $\phi \in A_q(\Delta, \Gamma)$ and $\psi \in B_q(\Delta, \Gamma)$, we define the Petersson inner product by

$$(\phi, \psi) = \iint_{\Delta/\Gamma} \lambda(z)^{2-2q}\phi(z)\overline{\psi(z)}\,dx\,dy, \qquad q \geq 2.$$

For $q = 1$ we shall interpret $A_1(\Delta, \Gamma)$ and $B_1(\Delta, \Gamma)$ as the Hilbert space of square integrable automorphic forms of weight (-2) with inner product defined by

$$(\phi, \psi) = \iint_{\Delta/\Gamma} \phi(z)\overline{\psi(z)}\,dx\,dy.$$

A holomorphic function E on Δ is called a holomorphic Eichler integral of order $(1 - q)$ on Δ if $E(Az)A'(z)^{1-q} - E(z) \in \Pi_{2q-2}$ on Δ, for all $A \in \Gamma$, where Π_{2q-2} is the vector space of polynomials of degree at most $2q - 2$. We shall say that an Eichler integral E of order $1 - q$ is bounded if $\phi = D^{2q-1}E \in B_q(\Delta, \Gamma)$, where D means differentiation with respect to z. The projection of ϕ to Δ/Γ is then a meromorphic q-differential on $\overline{\Delta/\Gamma}$ with order $\geq -(q - 1)$ at the punctures on Δ/Γ. An Eichler integral E on Δ is called quasi-bounded if the projection of $D^{2q-1}E$ to Δ/Γ can be extended as a meromorphic q-differential to $\overline{\Delta/\Gamma}$ whose order at a puncture is $\geq -q$. The space of bounded Eichler integrals modulo Π_{2q-2} will be denoted by $PE_{1-q}(\Delta, \Gamma)$. Similarly $E_{1-q}(\Delta, \Gamma)$ denotes the space of quasi-bounded Eichler integrals modulo Π_{2q-2}.

Let $f \in E_{1-q}(\Delta, \Gamma)$ and E a representative of f and set $D^{2q-1}E = \phi$. We set

$$f_{n-j}(z) = \sum_{k=0}^{j} ((-1)^k j!/(j-k)!)\, z^{j-k} D^{2q-2-k} E(z)$$

and set

$$(1) \qquad \mathfrak{f}(z) = \begin{pmatrix} f_0(z) \\ f_1(z) \\ \vdots \\ f_n(z) \end{pmatrix}, \quad I' = \begin{pmatrix} 1 & & & & & 0 \\ & -{}_nC_1 & & & & \\ & & {}_nC_2 & & & \\ & & & \ddots & & \\ & & & & -{}_nC_{n-1} & \\ 0 & & & & & 1 \end{pmatrix}$$

$\mathfrak{F}(z) = I'\widetilde{\mathfrak{f}(z)}$ and $\omega(z) = \phi(z)\binom{z}{1}^n dz$, where ${}_nC_i = n!/(n-1)!i!$. We call $\mathfrak{f}(z)$ and $\mathfrak{F}(z)$ column function vectors of length $n + 1$ associated with E. For each $A \in \Gamma$ we define X_A and P_A by

$$X_A = \mathfrak{f}(Az) - M(A)\mathfrak{f}(z)$$

and

$$P_A = {}^tM(A)\mathfrak{F}(Az) - \mathfrak{F}(z)$$

and denote them by $\mathrm{pd}_A(\mathfrak{f})$ and $\mathrm{pd}_A(\mathfrak{F})$, respectively. We call X_A and P_A periods of $\mathfrak{f}$ and $\mathfrak{F}$ for $A \in \Gamma$, respectively. The mapping $A \mapsto P_A$ satisfies $P_{AB} = {}^tM(B)P_A + P_B$ for any $A, B \in \Gamma$, as is easily seen. Then a cohomology class is defined, which depends only on f and not E. We define by $E_{1-q}(\Delta, \Gamma, M)$ the space of all $\mathfrak{F}(z)$ modulo $\mathbf{C}^{2q-1}$. Similarly we define $PE_{1-q}(\Delta, \Gamma, M)$. Thus by the obvious way we may define a mapping

$$\alpha: E_{1-q}(\Delta, \Gamma, M) \to H^1(\Gamma, \mathbf{C}^{2q-1}, M)$$

and we know that $\alpha(PE_{1-q}(\Delta, \Gamma, M)) \subset PH^1(\Gamma, \mathbf{C}^{2q-1}, M)$ by the method similar to that of Kra [6].

If $a_1, a_2, \cdots, a_{2q-1}$ are distinct points in Λ, and $\psi \in B_q(\Delta, \Gamma)$, then we call

$$\frac{(z-a_1)\cdots(z-a_{2q-1})}{2\pi i}\iint_{\Omega}\frac{\lambda(\zeta)^{2-2q}\overline{\psi(\zeta)}\,d\zeta \wedge d\bar{\zeta}}{(\zeta - z)(\zeta - a_1)\cdots(\zeta - a_{2q-1})},$$

$z \in \mathbf{C}$, $q \geq 2$, a potential for ψ, and denote it by $\mathrm{Pot}(\psi)$. For $A \in \Gamma$, we define a period of potential of $\mathrm{Pot}(\psi)$ by setting

$$\mathrm{pd}_A\, \mathrm{Pot}(\psi)(z) = \mathrm{Pot}(\psi)(Az)A'(z)^{1-q} - \mathrm{Pot}(\psi)(z), \qquad z \in \mathbf{C}.$$

It is easily seen that $\mathrm{Pot}(\psi)\,|\,\Omega - \Delta \in PE_{1-q}(\Omega - \Delta, \Gamma)$ for $\psi \in B_q(\Delta, \Gamma)$. We set

$$g_{n-j}(z) = \sum_{k=0}^{j}((-1)^k j!/(j-k)!)z^{j-k}D^{2q-2-k}\,\mathrm{Pot}(\psi)(z), \qquad z \in \Omega - \Delta.$$

We set

$$\mathfrak{g}(z) = \begin{pmatrix} g_0(z) \\ g_1(z) \\ \vdots \\ g_n(z) \end{pmatrix} \tag{2}$$

and set $\mathfrak{G}(z) = I'\widetilde{\mathfrak{g}(z)}$. We call $\mathfrak{g}(z)$ and $\mathfrak{G}(z)$ column function vectors of length $n+1$ associated with $\mathrm{Pot}(\psi)$.

For each $A \in \Gamma$, we define Y_A and Q_A by

$$Y_A = \mathfrak{g}(Az) - M(A)\mathfrak{g}(z), \quad z \in \Omega - \Delta,$$

and

$$Q_A = {}^tM(A)\mathfrak{G}(Az) - \mathfrak{G}(z), \quad z \in \Omega - \Delta,$$

and denote them by $\mathrm{pd}_A(\mathfrak{g})$ and $\mathrm{pd}(\mathfrak{G})$, respectively. We call Y_A and Q_A periods of $\mathfrak{g}$ and $\mathfrak{G}$ for $A \in \Gamma$, respectively. The mapping $A \mapsto Q_A$ satisfies $Q_{AB} = {}^tM(B)Q_A + Q_B$, for any $A, B \in \Gamma$, as is easily seen. Then a cohomology class is defined, which depends only on ψ. The definition of Y_A, Q_A, etc., applies to the case $\Omega - \Delta \neq \emptyset$. These functions for the remaining case will be defined in the remark after Lemma 4. Similarly as above we define

$$\beta^*: B_q(\Delta, \Gamma) \to H^1(\Gamma, \mathbf{C}^{2q-1}, M).$$

Let Γ_1 be a subgroup of Γ which keeps Δ_1 invariant, and let h be a conformal mapping from Δ_1 on the upper half-plane U. Set $\Gamma_1' = h\Gamma_1 h^{-1}$. Then Γ_1' is a fuchsian group of the first kind. Let ω_0 be the fundamental region defined by Shimura [7] for Γ_1' in U, then we let its boundary consist of sides $E_{A_1'}, E_{B_1'}, -A_1'(E_{A_1'}), -B_1'^{-1}(E_{B_1'}), \cdots, E_{A_g'}, E_{B_g'}, -A_g'(E_{A_g'}), -B_g'^{-1}(E_{B_g'}), E_{C_1'}, -C_1'(E_{C_1'}), \cdots, E_{C_j'}, -C_j'(E_{C_j'}), E_{D_1'}, -D_1'(E_{D_1'}), \cdots, E_{D_k'}, -D_k'(E_{D_k'})$, where $A_\lambda', B_\lambda', C_\mu'$ and D_ν' are generators of Γ_1' with relations $D_k \cdots D_1 C_j \cdots C_1 \prod_{\lambda=1}^g B_\lambda'^{-1}A_\lambda'^{-1}B_\lambda' A_\lambda' = 1$ and $C_\mu'^{e_\mu} = 1$ $(\mu = 1, 2, \cdots, j)$. If we set $A_\lambda = h^{-1}A_\lambda' h$, $B_\lambda = h^{-1}B_\lambda' h$, $C_\mu = h^{-1}C_\mu' h$ and $D_\nu = h^{-1}D_\nu' h$ $(\lambda = 1, \cdots, g;\ \mu = 1, \cdots, j;\ \nu = 1, \cdots, k)$, then Γ_1 is generated by $A_\lambda, B_\lambda, C_\mu$ and D_ν with relations $D_k \cdots D_1 C_j \cdots C_1 \prod_{\lambda=1}^g B_\lambda^{-1}A_\lambda^{-1}B_\lambda A_\lambda = 1$ and $C_\mu^{e_\mu} = 1$ $(\mu = 1, \cdots, j)$. We set $\omega_0 = h^{-1}(\omega_0')$ and set $E_{A_\lambda} = h^{-1}(E_{A_\lambda'})$, $E_{B_\lambda} = h^{-1}(E_{B_\lambda'})$, $E_{C_\mu} = h^{-1}(E_{C_\mu'})$ and $E_{D_\nu} = h^{-1}(E_{D_\nu'})$. Then we easily see that ω_0 is a fundamental region for Γ_1 in Δ_1 whose boundary consists of $E_{A_1}, E_{B_1}, -A_1(E_{A_1}), -B_1^{-1}(E_{B_1}), \cdots, E_{A_g}, E_{B_g}, -A_g(E_{A_g}), -B_g^{-1}(E_{B_g}), E_{C_1}, -C_1(E_{C_1}), \cdots, E_j, -C_j(E_{C_j}), E_{D_1}, -D_1(E_{D_1}), \cdots, E_{D_k}, -D_k(E_{D_k})$, since $-A_\lambda(E_{A_\lambda}) = h^{-1}(A_\lambda'(E_{A_\lambda'}))$, $-B_\lambda(E_{B_\lambda}) = h^{-1}(-B_\lambda'(E_{B_\lambda'}))$, $-C_\mu(E_{C_\mu}) = h^{-1}(-C_\mu'(E_{C_\mu'}))$ and $-D_\nu(E_{D_\nu}) = h^{-1}(-D_\nu'(E_{D_\nu'}))$. Since h is a conformal mapping, the sides of ω_0 consist of piecewise analytic arcs. We set $S_\lambda = B_\lambda^{-1}A_\lambda^{-1}B_\lambda A_\lambda$ and $T_\lambda = S_\lambda \cdots S_1$ $(\lambda = 1, \cdots, g)$.

2. **Lemmas.** In this section we state some lemmas which are necessary to prove the subsequent theorems. Many of the properties in Lemmas 2, 3, 4 and 5 below can be sumarized by saying that there is an isomorphism $\Pi_{2q-2} \to \mathbf{C}^{2q-1}$ which commutes with the action of Γ. However we shall state them for the sake of later use. For each $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$, we denote by $A(z) = (az+b)/(cz+d)$. We set $n = 2q - 2$ once and for all.

Lemma 1. *For* $A \in \Gamma$

$$M(A) = I'^{-1}(\widetilde{{}^tM(A)})^{-1}I'.$$

Proof. We set $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $ad - bc = 1$. The (k, l) element of $M(A)$ is

$$\sum_{i+j=l-1} {}_{2q-1-k}C_i \; {}_{k-1}C_j \, a^{2q-k-i-1} c^{k-1-j} b^i d^j.$$

The (k, l) element of $M(A)$ is the (l, k) element of ${}^tM(A)$, which is the $(2q-l, 2q-k)$ element of ${}^tM(A)$. Hence the $(2q-l, 2q-k)$ element of ${}^tM(A)I'$ is

$$(3)\quad (-1)^{2q-k-1}\; {}_{2q-2}C_{2q-k-1} \sum_{i+j=l-1} {}_{2q-1-k}C_i \; {}_{k-1}C_j \, a^{2q-1-k-i} c^{k-1-j} b^i d^j.$$

On the other hand the $(2q-l, 2q-k)$ element of $I'M(A^{-1})$ is

$$(4)\quad (-1)^{2q-l-1}\; {}_{2q-2}C_{2q-l-1} \sum_{i+j=2q-1-k} {}_{l-1}C_i \; {}_{2q-l-1}C_j \, d^{l-1-i} (-c)^{2q-l-1-j} (-b)^i a^j.$$

We easily see that (3) and (4) are the same, that is,

$$(\widetilde{{}^tM(A)})I' = I'M(A^{-1}).$$

For the proof of some properties in Lemmas 2 through 4 below, see Shimura [7].

Lemma 2. *For* $A, B \in \Gamma$,

(1) $\begin{pmatrix} Az \\ 1 \end{pmatrix}^n A'(z)^{1-q} = M(A)\begin{pmatrix} z \\ 1 \end{pmatrix}^n$,

(2) $M(AB) = M(A)M(B)$,

(3) $M(A^{-1}) = M(A)^{-1}$.

Lemma 3. *Let* $f \in E_{1-q}(\Delta, \Gamma)$ *and* E *a representative of* f, *and* $\psi \in B_q(\Delta, \Gamma)$. *Let* $\mathfrak{f}(z)$, $\omega(z)$, $\mathfrak{g}(z)$, X_A, Y_A, P_A *and* Q_A *be defined as in* §1 *with respect to* E *and* ψ. *Then, for* $A, B \in \Gamma$,

(1) $\omega(Az) = M(A)\omega(z)$,

(2) $d\mathfrak{f}(z) = \omega(z)$,

(3) $E(z) = (1/n!)\, {}^t\mathfrak{f}(z) I' \begin{pmatrix} 1 \\ z \end{pmatrix}^n$, $z \in \Delta$,

(4) $E(Az)A'(z)^{1-q} - E(z) = (1/n!)\, {}^tP_A \begin{pmatrix} z \\ 1 \end{pmatrix}^n$, $z \in \Delta$,

(5) $X_{AB} = X_A + M(A)X_B$,

(6) $\mathrm{Pot}(\psi)(z) = (1/n!)\, {}^t\mathfrak{g}(z) I' \begin{pmatrix} 1 \\ z \end{pmatrix}^n$, $z \in \mathbf{C}$,

(7) $\mathrm{Pot}(\psi)(Az)A'(z)^{1-q} - \mathrm{Pot}(\psi)(z) = (1/n!)\, {}^tQ_A \begin{pmatrix} z \\ 1 \end{pmatrix}^n$, $z \in \Omega - \Delta$,

(8) $Y_{AB} = Y_A + M(A)Y_B$,

(9) $P_A = I'\widetilde{M(A)}^{-1}\mathfrak{X}_A$, $X_A = M(A)I'^{-1}\mathfrak{P}_A$,

(10) $Q_A = I'\widetilde{M(A)}^{-1}\mathfrak{Y}_A$, $Y_A = M(A)I'^{-1}\mathfrak{Q}_A$.

By (3) of Lemma 3 we have

Lemma 4. *For each* $A \in \Gamma$, X_A, P_A, Y_A *and* Q_A *are all number vectors of length* $2q-1$.

This means that

$$\mathrm{Pot}(\psi)(Az)A'(z)^{1-q} - \mathrm{Pot}(\psi)(z) = (1/n!)\, {}^tQ_A\binom{z}{1}^n$$

for $z \in \mathbf{C}$ and $A \in \Gamma$.

Remark. The above Q_A is defined in the case of $\Omega \neq \Delta$. However, since $\mathrm{Pot}(\psi)(Az)A'(z)^{1-q} - \mathrm{Pot}(\psi)(z) = v_A(z)$, $v_A \in \Pi_{2q-2}$, $A \in \Gamma$, $z \in \mathbf{C}$, we may define Q'_A by $v(A) = (1/n!)\, {}^tQ'_A\binom{z}{1}^n$ with $Q'_A \in \mathbf{C}^{2q-1}$. Then we easily see that $Q'_{AB} = {}^tM(B)Q'_A + Q'_B$, $A, B \in \Gamma$. We set $Y'_A = M(A)I'^{-1}\widetilde{Q}'_A$. Hereafter we take Q_A to be Q_A and Y_A to be Y'_A, and note that these definitions agree with previous ones and are valid in the case $\Omega = \Delta$ as well.

Noting the fact pointed out in the first part of this section, we have the following from Kra's decomposition theorem [5], [6].

Lemma 5. (1) $H^1(\Gamma, \mathbf{C}^{2q-1}, M) = \alpha(E_{1-q}(\Delta, \Gamma, M) + \beta^*(B_q(\Delta, \Gamma))$; (2) $\beta^*(B_q(\Delta, \Gamma)) \subset PH^1_\Delta(\Gamma, \mathbf{C}^{2q-1}, M)$.

3. The main theorem.

Theorem 1. *Let* Γ *be a nonelementary finitely generated Kleinian group and* Δ_1 *a simply connected component of* Ω *and set* $\Delta = \bigcup_{A \in \Gamma} A(\Delta_1)$. *Let* $f \in E_{1-q}(\Delta, \Gamma)$, E *an arbitrary representative of* f *and set* $D^{2q-1}_{\bar{z}}E = \phi$, $q \geq 2$. *Let* $\psi \in B_q(\Delta, \Gamma)$. *Let* $\mathfrak{f}(z)$ *and* $\mathfrak{g}(z)$ *be column function vectors* (1) *and* (2) *associated with* E *and* $\mathrm{Pot}(\psi)$, *respectively, and set* $\mathfrak{G}(z) = I'\widetilde{\mathfrak{g}(z)}$. *Let* $\mathrm{pd}_A\, \mathfrak{f}(z) = X_A$ *and* $\mathrm{pd}_A\, \mathfrak{G}(z) = Q_A$ *for each* $A \in \Gamma$. *Then*

$$\begin{aligned}
2in!(\phi, \psi) = {} & \sum_{\lambda=1}^{g} {}^tQ_{A_\lambda}[X_{A_\lambda^{-1}B_\lambda A_\lambda T_{\lambda-1}} - X_{T_{\lambda-1}}] \\
& + \sum_{\lambda=1}^{g} {}^tQ_{B_\lambda^{-1}}[X_{B_\lambda A_\lambda T_{\lambda-1}} - X_{A_\lambda^{-1}B_\lambda A_\lambda T_{\lambda-1}}] \\
& + \sum_{\mu=1}^{j} {}^tQ_{C_\mu}\left[e_\mu^{-1} \sum_{m=1}^{e_\mu - 1} X_{C_\mu^m} - X_{C_{\mu-1}\cdots C_1 T_g}\right] \\
& - \sum_{\nu=1}^{k} {}^tQ_{D_\nu}[X_{D_{\nu-1}\cdots D_1 C_j \cdots C_1 T_g}] + \sum_{\nu=1}^{k} {}^tQ_{D_\nu}\mathfrak{f}(s_\nu),
\end{aligned} \tag{I}$$

where s_ν *is a cusp point of* D_ν $(\nu = 1, 2, \cdots, k)$.

Proof. By (7) of Lemma 3 and Lemma 4, for every $A \in \Gamma$,

$$(5) \qquad \mathrm{Pot}(\psi)(Az)A'(z)^{1-q} - \mathrm{Pot}(\psi)(z) = (1/n!)\, {}^tQ_A \begin{pmatrix} z \\ 1 \end{pmatrix}^n, \qquad z \in \mathbf{C}.$$

Let $\eta(z)$ be a C^∞-function on Δ defined by Kra [5], [6], that is, (1) $0 \leq \eta \leq 1$; (2) for each $z \in \Delta$, there is a neighborhood $U(z)$ of z and a finite subset J of Γ such that $\eta(A(U(z))) = 0$ for each $A \notin J$; (3) $\sum_{\gamma \in \Gamma} \eta(\gamma z) = 1$, $z \in \Delta$; and (4) if $U_A \in \omega_0$ is a cusped region belonging to a puncture on Δ/Γ and A is the corresponding transformation, then $\eta \mid B(U_A) = 0$ all $B \in \Gamma - \{1, A\}$.

We set

$$\Theta(z) = \begin{pmatrix} \theta_0(z) \\ \theta_1(z) \\ \vdots \\ \theta_n(z) \end{pmatrix} = -\frac{1}{n!} \sum_{\gamma \in \Gamma} \eta(\gamma z) Q_\gamma + \frac{1}{n!} \sum_{\nu=1}^{k} \chi_\nu(z) \left\{ \sum_{\gamma \in \Gamma} \eta(\gamma z)\, {}^tM(\gamma) V_\nu \right\},$$

where V_ν is defined by ${}^tM(D_\nu)V_\nu - V_\nu = P_{D_\nu}$ $(\nu = 1, \cdots, k)$ and we let $\chi_\nu \in C^\infty(\Delta, \Gamma)$ be such that $0 \leq \chi_\nu \leq 1$, $\chi_\nu = 1$ in U_{D_ν} and $\chi_\nu = 0$ in $\bigcup_{i \neq \nu} U_{D_i}$ $(\nu = 1, \cdots, k)$. Then

$$\begin{aligned}
&{}^tM(A)\Theta(Az) - \Theta(z) \\
&= -\frac{1}{n!}\, {}^tM(A) \sum_{\gamma \in \Gamma} \eta(\gamma A z) Q_\gamma + \frac{1}{n!}\, {}^tM(A) \sum_{\nu=1}^{k} \chi_\nu(Az) \left(\sum_{\gamma \in \Gamma} \eta(\gamma A z)\, {}^tM(\gamma) V_\nu \right) \\
&\quad + \frac{1}{n!} \sum_{\gamma \in \Gamma} \eta(\gamma z) Q_\gamma - \frac{1}{n!} \sum_{\nu=1}^{k} \chi_\nu(z) \left(\sum_{\gamma \in \Gamma} \eta(\gamma z)\, {}^tM(\gamma) V_\nu \right) \\
&= -\frac{1}{n!} \sum_{\gamma \in \Gamma} \eta(\gamma A z)(Q_{\gamma A} - Q_A) + \sum_{\gamma \in \Gamma} \frac{1}{n!} \eta(\gamma z) Q_\gamma \\
&\quad + \frac{1}{n!} \sum_{\nu=1}^{k} \chi_\nu(z) \left(\sum_{\gamma \in \Gamma} \eta(\gamma A z)\, {}^tM(\gamma A) V_\nu \right) \\
&\quad - \frac{1}{n!} \sum_{\nu=1}^{k} \chi_\nu(z) \left(\sum_{\gamma \in \Gamma} \eta(\gamma z)\, {}^tM(\gamma) V_\nu \right) \\
&= \frac{1}{n!} Q_A,
\end{aligned}$$

for each $A \in \Gamma$.

We set

$$\mu(z) = \begin{pmatrix} \mu_0(z) \\ \mu_1(z) \\ \vdots \\ \mu_n(z) \end{pmatrix} = \begin{pmatrix} \partial\theta_0(z)/\partial\bar{z} \\ \partial\theta_1(z)/\partial\bar{z} \\ \vdots \\ \partial\theta_n(z)/\partial\bar{z} \end{pmatrix}.$$

Then we easily see that, for $A \in \Gamma$,

$$^tM(A)\mu(Az)\overline{A'(z)} = \mu(z), \qquad z \in \Delta.$$

Thus for $\phi = D^{2q-1}E$,

$$\iint_{\Delta/\Gamma} \phi(z)\ {}^t\mu(z)\binom{z}{1}^n dz \wedge d\bar{z}$$

is well defined.

Next we show that

$$(6) \qquad \iint_{\omega_0} \phi(z)\ {}^t\mu(z)\binom{z}{1}^n dz \wedge d\bar{z} - \iint_{\omega_0} \phi(z)\lambda(z)^{2-2q}\ \overline{\psi(z)}\, dz \wedge d\bar{z} = 0.$$

For, since for $A \in \Gamma$,

$$^t\Theta(Az)\binom{Az}{1}^n A'(z)^{1-q} - {}^t\Theta(z)\binom{z}{1}^n = \frac{1}{n!}\, {}^tQ_A\binom{z}{1}^n, \qquad z \in \Delta,$$

by (5) we have

$$(7) \qquad \left\{{}^t\Theta(Az)\binom{Az}{1}^n - \operatorname{Pot}(\psi)(Az)\right\} A'(z)^{1-q} = {}^t\Theta(z)\binom{z}{1}^n - \operatorname{Pot}(\psi)(z),$$

$z \in \Delta$. By using Stokes' theorem after Bers' trick [3], (6) is equal to

$$(8) \qquad \int_{\partial\omega_0} \left\{{}^t\Theta(z)\binom{z}{1}^n - \operatorname{Pot}(\psi)(z)\right\}\phi(z)\, dz.$$

This is equal to zero, in fact, since its integrals along two identified sides cancel each other and, therefore, (8) is equal to zero, that is,

$$-2i(\phi, \psi) = \iint_{\omega_0} \phi(z)\ {}^t\mu(z)\binom{z}{1}^n dz \wedge d\bar{z}.$$

On the other hand,

$$\iint_{\omega_0} {}^t\mu(z)\phi(z)\binom{z}{1}^n dz \wedge d\bar{z} = \int_{\partial\omega_0} {}^t\Theta(z)\binom{z}{1}^n \phi(z)\,dz = \int_{\partial\omega_0} {}^t\Theta(z)\omega(z),$$

where $\omega(z) = \phi(z)\binom{z}{1}^n dz$. Then

$$\int_{\partial\omega_0} {}^t\Theta(z)\omega(z) = \int_{\partial\omega_0} {}^t\Theta(z)\,d\mathfrak{f}(z)$$

$$= \sum_{\lambda=1}^{g}\left(\int_{E_{A_\lambda}} {}^t\Theta(z)\,d\mathfrak{f}(z) - \int_{A_\lambda(E_{A_\lambda})} {}^t\Theta(z)\,d\mathfrak{f}(z)\right)$$

$$+ \sum_{\lambda=1}^{g}\left(\int_{E_{B_\lambda}} {}^t\Theta(z)\,d\mathfrak{f}(z) - \int_{B_\lambda^{-1}(E_{B_\lambda})} {}^t\Theta(z)\,d\mathfrak{f}(z)\right)$$

$$+ \sum_{\mu=1}^{j}\left(\int_{E_{C_\mu}} {}^t\Theta(z)\,d\mathfrak{f}(z) - \int_{C_\mu(E_{C_\mu})} {}^t\Theta(z)\,d\mathfrak{f}(z)\right)$$

$$+ \sum_{\nu=1}^{k}\left(\int_{E_{D_\nu}} {}^t\Theta(z)\,d\mathfrak{f}(z) - \int_{D_\nu(E_{D_\nu})} {}^t\Theta(z)\,d\mathfrak{f}(z)\right).$$

Now for any element A of $\{A_\lambda, B_\lambda^{-1}, C_\mu, D_\nu\ (\lambda = 1, \cdots, g;\ \mu = 1, \cdots, j;\ \nu = 1, \cdots, k)\}$,

$$\int_{A(E_A)} {}^t\Theta(z)\,d\mathfrak{f}(z) = \int_{E_A} {}^t\Theta(A(z))\,d\mathfrak{f}(A(z))$$

$$= \int_{E_A}\left\{{}^t({}^tM(A^{-1})\Theta(z)) + \frac{1}{n!}\,{}^t({}^tM(A^{-1})Q_A)\,d(M(A)\mathfrak{f}(z) + X_A)\right\}$$

$$= \int_{E_A} {}^t\Theta(z)\,d\mathfrak{f}(z) + \int_{E_A}\frac{1}{n!}\,{}^tQ_A\,d\mathfrak{f}(z),$$

so that

$$\int_{\partial\omega_0} {}^t\Theta(z)\omega(z) = -\left\{\sum_{\lambda}\left(\int_{E_{A_\lambda}}\frac{1}{n!}\,{}^tQ_A\,d\mathfrak{f}(z) + \int_{E_{B_\lambda}}\frac{1}{n!}\,{}^tQ_{B_\lambda^{-1}}\,d\mathfrak{f}(z)\right)\right.$$

$$\left. + \sum_{\mu}\int_{E_{C_\mu}}\frac{1}{n!}\,{}^tQ_{C_\mu}\,d\mathfrak{f}(z) + \sum_{\nu}\int_{E_{D_\nu}}\frac{1}{n!}\,{}^tQ_{D_\nu}\,d\mathfrak{f}(z)\right\}.$$

Denote by u_0 the starting point of E_{A_1}. Then

$$2in!(\phi,\psi)=\sum_{\lambda=1}^{g}{}^tQ_{A_\lambda}[M(A_\lambda^{-1}B_\lambda A_\lambda T_{\lambda-1})-M(T_{\lambda-1})]\mathfrak{f}(u_0)$$

$$+\sum_{\lambda=1}^{g}{}^tQ_{B_\lambda^{-1}}[M(B_\lambda A_\lambda T_{\lambda-1})-M(A_\lambda^{-1}B_\lambda A_\lambda T_{\lambda-1})]\mathfrak{f}(u_0)$$

$$+\sum_{\mu=1}^{j}{}^tQ_{C_\mu}[M(C_{\mu-1}\cdots C_1T_g)]\mathfrak{f}(u_0)$$

(9)
$$-\sum_{\nu=1}^{k}{}^tQ_{D_\nu}[M(D_{\nu-1}\cdots D_1C_j\cdots C_1T_g)]\mathfrak{f}(u_0)$$

$$+\sum_{\lambda=1}^{g}{}^tQ_{A_\lambda}[X_{A_\lambda^{-1}B_\lambda A_\lambda T_{\lambda-1}}-X_{T_{\lambda-1}}]$$

$$+\sum_{\lambda=1}^{g}{}^tQ_{B_\lambda^{-1}}[X_{B_\lambda A_\lambda T_{\lambda-1}}-X_{A_\lambda^{-1}B_\lambda A_\lambda T_{\lambda-1}}]$$

$$+\sum_{\mu=1}^{j}{}^tQ_{C_\mu}\left[e_\mu^{-1}\sum_{m=1}^{e_\mu-1}X_{C_\mu^m}-X_{C_{\mu-1}\cdots C_1T_g}\right]$$

$$-\sum_{\nu=1}^{k}{}^tQ_{D_\nu}[X_{D_{\nu-1}\cdots D_1C_j\cdots C_1T_g}]+\sum_{\nu=1}^{k}{}^tQ_{D_\nu}\mathfrak{f}(s_\nu).$$

For, let t_μ be a fixed point for C_μ $(\mu=1,\cdots,j)$ in $\bar{\omega}_0$. Then

$$\mathfrak{f}(t_\mu)=\mathfrak{f}(C_\mu^m(t_\mu))=M(C_\mu^m)\mathfrak{f}(t_\mu)+X_{C_\mu^m}$$

and

$$e_\mu\mathfrak{f}(t_\mu)=\sum_{m=0}^{e_\mu-1}M(C_\mu^m)\mathfrak{f}(t_\mu)+\sum_{m=0}^{e_\mu-1}X_{C_\mu^m}.$$

On the other hand

$$0=Q_{C_\mu^{e_\mu}}=\sum_{m=0}^{e_\mu-1}{}^tM(C_\mu^m)Q_{C_\mu}.$$

Hence

$$\sum_{\mu} {}^tQ_{C_\mu}\mathfrak{f}(t_\mu) = \sum_{\mu} {}^tQ_{C_\mu} e_\mu^{-1} \left\{ \sum_{m=0}^{e_\mu - 1} M(C_\mu^m)\mathfrak{f}(t_\mu) + \sum_{m=0}^{e_\mu - 1} X_{C_\mu^m} \right\}$$

$$= \sum_{\mu} e_\mu^{-1} \left\{ \sum_{m=0}^{e_\mu - 1} {}^tQ_{C_\mu} M(C_\mu^m)\mathfrak{f}(t_\mu) + {}^tQ_{C_\mu} \sum_{m=0}^{e_\mu - 1} X_{C_\mu^m} \right\}$$

$$= \sum_{\mu} {}^tQ_{C_\mu} \left(\sum_{m=0}^{e_\mu - 1} e_\mu^{-1} X_{C_\mu^m} \right) + \sum_{\mu} e_\mu^{-1} \left(\sum_{m=0}^{e_\mu - 1} {}^tQ_{C_\mu} M(C_\mu^m)\mathfrak{f}(t_\mu) \right)$$

$$= \sum_{\mu} {}^tQ_{C_\mu} e_\mu^{-1} \left(\sum_{m=0}^{e_\mu - 1} X_{C_\mu^m} \right).$$

We have

$$\sum_{\mu} {}^tQ_{C_\mu}\mathfrak{f}(C_{\mu-1} \cdots C_1 T_g(u_0))$$

$$= \sum_{\mu} {}^tQ_{C_\mu} M(C_{\mu-1} \cdots C_1 T_g)\mathfrak{f}(u_0) + \sum_{\mu} {}^tQ_{C_\mu} X_{C_{\mu-1} \cdots C_1 T_g}.$$

Furthermore we have

$$\sum_{\nu} {}^tQ_{D_\nu}(\mathfrak{f}(s_\nu) - \mathfrak{f}(D_{\nu-1} \cdots D_1 C_j \cdots C_1 T_g(u_0))$$

$$= \sum_{\nu} {}^tQ_{D_\nu}\mathfrak{f}(s_\nu) - \sum_{\nu} {}^tQ_{D_\nu} M(D_{\nu-1} \cdots D_1 C_j \cdots C_1 T_g)\mathfrak{f}(u_0)$$

$$- \sum_{\nu} {}^tQ_{D_\nu} X_{D_{\nu-1} \cdots D_1 C_j \cdots C_1 T_g}.$$

We denote by Ψ the sum of the first four terms of (9). Set

$$N_1 = \sum_{\lambda=1}^{g} {}^t(M(A_\lambda^{-1} B_\lambda A_\lambda T_{\lambda-1}) - M(T_{\lambda-1}))Q_{A_\lambda}$$

$$+ \sum_{\lambda=1}^{g} {}^t(M(B_\lambda A_\lambda T_{\lambda-1}) - M(A_\lambda^{-1} B_\lambda A_\lambda T_{\lambda-1}))Q_{B_\lambda^{-1}},$$

$$N_2 = \sum_{\mu=1}^{j} {}^tM(C_{\mu-1} \cdots C_1 T_g)Q_{C_\mu}, \quad \text{and}$$

$$N_3 = \sum_{\nu=1}^{k} {}^tM(D_{\nu-1} \cdots D_1 C_j \cdots C_1 T_g)Q_{D_\nu}.$$

Then we have

$$N_1 = \sum_{\lambda=1}^{g} (Q_{T_{\lambda-1}} - Q_{T_\lambda}),$$

$$N_2 = \sum_{\mu=1}^{j} (Q_{C_\mu C_{\mu-1}\cdots C_1 T_g} - Q_{C_{\mu-1}\cdots C_1 T_g}), \quad \text{and}$$

$$N_3 = \sum_{\nu=1}^{k} (Q_{D_\nu \cdots D_1 C_j \cdots C_1 T_g} - Q_{D_{\nu-1}\cdots D_1 C_j \cdots C_1 T_g}),$$

hence $N_1 - N_2 - N_3 = 0$, since $D_k \cdots D_1 C_j \cdots C_1 T_g = 1$, that is, $Q_{D_k \cdots D_1 C_j \cdots C_1 T_g} = 0$. Thus we have the desired result. Our proof is now complete.

Corollary 1 (Period inequality). *Under the same assumptions as in Theorem* 1 (*let* $\phi = \psi$),

$$\frac{1}{2i}\Bigg\{\sum_{\lambda=1}^{g} {}^tQ_{A_\lambda}[X_{A_\lambda^{-1} B_\lambda A_\lambda T_{\lambda-1}} - X_{T_{\lambda-1}}] + \sum_{\lambda=1}^{g} {}^tQ_{B_\lambda^{-1}}[X_{B_\lambda A_\lambda T_{\lambda-1}} - X_{A_\lambda^{-1} B_\lambda A_\lambda T_{\lambda-1}}]$$

$$+ \sum_{\mu=1}^{j} {}^tQ_{C_\mu}\left[e_\mu^{-1} \sum_{m=1}^{e_\mu - 1} X_{C_\mu^m} - X_{C_{\mu-1}\cdots C_1 T_g}\right]$$

$$- \sum_{\nu=1}^{k} {}^tQ_{D_\nu}[X_{D_{\nu-1}\cdots D_1 C_j \cdots C_1 T_g}] + \sum_{\nu=1}^{k} {}^tQ_{D_\nu}\mathfrak{f}(s_\nu)\Bigg\} > 0.$$

Corollary 2. *Let* Γ *be a fuchsian group of the first kind, and let* $\Delta_1 = U$. *Let* $f \in PE_{1-q}(U, \Gamma)$, $f^* \in E_{1-q}(U, \Gamma)$, $q \geq 2$, *and* E, E^* *arbitrary representatives of* f *and* f^*, *respectively. Set* $D^{2q-1}E = \phi$ *and* $D^{2q-1}E^* = \phi^*$. *Let* $\mathfrak{f}$ *and* $\mathfrak{f}^*$ *be column vectors of length* $2q-1$ *of the form* (1) *associated with* E *and* E^*, *respectively, and set* $\mathrm{pd}_A\, \mathfrak{f}(z) = X_A$ *and* $\mathrm{pd}_A\, \mathfrak{f}^*(z) = X_A^*$ *for each* $A \in \Gamma$. *Then*

$$(\phi^*, \phi) = \frac{(-1)^{q-1}}{2i}\Bigg\{\sum_{\lambda=1}^{g} {}^t\tilde{\tilde{X}}_{A_\lambda} I'[X^*_{A_\lambda^{-1} B_\lambda A_\lambda T_{\lambda-1}} - X^*_{T_{\lambda-1}}]$$

$$\text{(II)} \qquad + \sum_{\lambda=1}^{g} {}^t\tilde{\tilde{X}}_{B_\lambda} I'[X^*_{B_\lambda A_\lambda T_{\lambda-1}} - X^*_{A_\lambda^{-1} B_\lambda A_\lambda T_{\lambda-1}}]$$

$$+ \sum_{\mu=1}^{j} {}^t\tilde{\tilde{X}}_{C_\mu^{-1}} I'\left[e_\mu^{-1} \sum_{m=1}^{e_\mu - 1} X^*_{C_\mu^m} - X^*_{C_{\mu-1}\cdots C_1 T_g}\right]$$

$$- \sum_{\nu-1}^{k} {}^t\tilde{\tilde{X}}_{D_\nu^{-1}} I'[X^*_{D_{\nu-1}\cdots D_1 C_j \cdots C_1 T_g}] + \sum_{\nu=1}^{k} {}^t\tilde{\tilde{X}}_{D_\nu^{-1}} I'\mathfrak{f}(s_\nu)\Bigg\}.$$

Proof. If we set

$$\psi(z) = \frac{(2q-1)!}{2\pi i} \iint_U \frac{(\zeta - \bar{\zeta}/2i)^{2q-2}\overline{\phi(\zeta)}\, d\zeta \wedge d\bar{\zeta}}{(\zeta - z)^{2q}}, \quad z \in L,$$

then $\psi \in B_q(L, \Gamma)$ (Bers [3]), where L is the lower half-plane. Set $E_1(z) \doteq \mathrm{Pot}(\phi)(z)$, $z \in L$. Then by Bers [3], $D^{2q-1}E_1 = \psi$ on L. If for $z \in U$, we set $E_2(z) = \overline{E_1(\bar{z})}$ and $D^{2q-1}E_2(z) = \psi_1(z)$; then $\psi_1(z) = \overline{\psi(\bar{z})}$, $\psi_1 \in B_q(U, \Gamma)$ and $E_2 \in E_{1-q}(U, \Gamma)$. Then $\overline{\psi(\bar{z})} = c_q \phi(z)$, $z \in U$ (see Kra [5, p. 554]), so that $\psi_1(z) = c_q \phi(z)$, $z \in U$, where $c_q = (-1)^{q-1}(2q-2)!$. Hence $E_2 = c_q E$.

On the other hand if for each $A \in \Gamma$, we set $\mathrm{pd}_A(\mathrm{Pot}(\phi)) = Q_A$ and $\mathrm{pd}_A(\mathfrak{F}) = P_A$, then

$$E_2(Az)A'(z)^{1-q} - E_2(z) = \overline{E_1(\overline{Az})A'(\bar{z})^{1-q} - E_1(\bar{z})}$$
$$= \frac{1}{n!}\, {}^t\bar{Q}_A \binom{z}{1}^n, \quad z \in U,$$

and

$$E(Az)A'(z)^{1-q} - E(z) = \frac{1}{n!}\, {}^tP_A \binom{z}{1}^n, \quad z \in U,$$

where $\mathfrak{F}(z) = I'\widetilde{\mathfrak{f}(z)}$. Hence $\bar{Q}_A = c_q P_A$. Using Lemma 3, we have

$${}^tP_A = {}^t(I'\widetilde{M(A^{-1})}\tilde{X}_A)$$
$$= {}^t\tilde{X}_A\, {}^t\widetilde{M(A)}^{-1} I'$$
$$= -\, {}^t\tilde{X}_{A^{-1}} I'.$$

Substituting these into (I), we have the desired result. Our proof is now complete.

Remark. By using the same method as in the proof of Theorem 2 below, we can show that even in the case of $q = 1$, the above corollary remains valid.

4. **Period relations and inequalities.** By modifying Shimura's method [7], we can prove the following theorem from Lemmas 1, 2, 3 and 4. The proof is omitted here.

Theorem 2 (Period relation). *Let Γ be the same group as in Theorem 1. Let $f_1, f_2 \in E_{1-q}(\Delta, \Gamma, M)$, $q \geq 1$ and $\mathfrak{F}_1, \mathfrak{F}_2$ arbitrary representatives of f_1 and f_2, respectively. Set $\mathrm{pd}_A\, \mathfrak{F}_1 = P_A^{(1)}$ and $\mathrm{pd}_A\, \mathfrak{F}_2 = P_A^{(2)}$ for each $A \in \Gamma$. Then*

$$\sum_{\lambda=1}^{g} {}^tP^{(1)}_{A_\lambda}[M(A_\lambda^{-1}B_\lambda A_\lambda T_{\lambda-1})I'^{-1}\tilde{P}^{(2)}_{A_\lambda^{-1}B_\lambda A_\lambda T_{\lambda-1}} - M(T_{\lambda-1})I'^{-1}\tilde{P}^{(2)}_{T_{\lambda-1}}]$$

$$+\sum_{\lambda=1}^{g} {}^tP^{(1)}_{B_\lambda^{-1}}[M(B_\lambda A_\lambda T_{\lambda-1})I'^{-1}\tilde{P}^{(2)}_{B_\lambda A_\lambda T_{\lambda-1}} - M(A_\lambda^{-1}B_\lambda A_\lambda T_{\lambda-1})I'^{-1}\tilde{P}^{(2)}_{A_\lambda^{-1}B_\lambda A_\lambda T_{\lambda-1}}]$$

$$+\sum_{\mu=1}^{j} {}^tP^{(1)}_{C_\mu}\left[e_\mu^{-1}\sum_{m=1}^{e_\mu-1} M(C_\mu^m)I'^{-1}\tilde{P}^{(2)}_{C_\mu^m} - M(C_{\mu-1}\cdots C_1T_g)I'^{-1}\tilde{P}^{(2)}_{C_{\mu-1}\cdots C_1T_g}\right]$$

$$-\sum_{\nu=1}^{k} {}^tP^{(1)}_{D_\nu}[M(D_{\nu-1}\cdots D_1C_j\cdots C_1T_g)I'^{-1}\tilde{P}^{(2)}_{D_{\nu-1}\cdots D_1C_j\cdots C_1T_g}]$$

$$+\sum_{\nu=1}^{k} {}^tP^{(1)}_{D_\nu}\mathfrak{f}(s_\nu) = 0.$$

Corollary 1 (Period relation for a fuchsian group). *Let Γ be a fuchsian group of the first kind, and let $\Delta_1 = U$. Let $f_1, f_2 \in E_{1-q}(\Delta, \Gamma, M)$, $q \geq 1$, and $\mathfrak{F}_1, \mathfrak{F}_2$ arbitrary representatives of f_1 and f_2, respectively and set $\mathfrak{F}_1 = I'\mathfrak{f}_1$ and $\mathfrak{F}_2 = I'\tilde{\mathfrak{f}}_2$. Set $\mathrm{pd}_A\ \mathfrak{f}_1 = X_A^{(1)}$ and $\mathrm{pd}_A\ \mathfrak{f}_2 = X_A^{(2)}$ for every $A \in \Gamma$. Then*

$$\text{(III)}\quad \begin{aligned}&\sum_{\lambda=1}^{g} {}^t\tilde{X}^{(1)}_{A_\lambda^{-1}}I'[X^{(2)}_{A_\lambda^{-1}B_\lambda A_\lambda T_{\lambda-1}} - X^{(2)}_{T_{\lambda-1}}]\\ &+\sum_{\lambda=1}^{g} {}^t\tilde{X}^{(1)}_{B_\lambda}I'[X^{(2)}_{B_\lambda A_\lambda T_{\lambda-1}} - X^{(2)}_{A_\lambda^{-1}B_\lambda A_\lambda T_{\lambda-1}}]\\ &+\sum_{\mu=1}^{j} {}^t\tilde{X}^{(1)}_{C_\mu^{-1}}I'\left[e_\mu^{-1}\sum_{m=1}^{e_\mu-1} X^{(2)}_{C_\mu^m} - X^{(2)}_{C_{\mu-1}\cdots C_1T_g}\right]\\ &-\sum_{\nu=1}^{k} {}^t\tilde{X}^{(1)}_{D_\nu^{-1}}I'[X^{(2)}_{D_{\nu-1}\cdots D_1C_j\cdots C_1T_g}] + \sum_{\nu=1}^{k} {}^t\tilde{X}^{(1)}_{D_\nu}I'\mathfrak{f}(s_\nu) = 0.\end{aligned}$$

Remark. Especially, when $q = 1$,

$$\sum_{\lambda=1}^{g} (X^{(1)}_{B_\lambda}X^{(2)}_{A_\lambda} - X^{(1)}_{A_\lambda}X^{(2)}_{B_\lambda}) = 0.$$

For $I' = 1$ and $X^{(i)}_{A^{-1}} = -X^{(i)}_A$ $(i = 1, 2)$ for all $A \in \Gamma$. This is the period relation for abelian integrals. We set $X_A = x_A + iy_A$, where x_A and y_A are real number vectors.

Corollary 2 (Period inequality). *Let Γ be the same group as in the above corollary. Let $f \in E_{1-q}(U, \Gamma, M)$ and $\mathfrak{F}$ be representative of f, $q \geq 1$. Set*

$\mathfrak{F} = l'\mathfrak{f}$, $(1/n!)D^{2q-1}\mathfrak{F}(z)\binom{z}{1}^n = \phi(z)$ *and* $\mathrm{pd}_A\ \mathfrak{f}(z) = x_A + iy_A$ *for every* $A \in \Gamma$.. *If* $\phi \neq 0$, *then*

$$(-1)^q\Bigg[\sum_{\lambda=1}^{g} {}^t\tilde{y}_{A_\lambda^{-1}} l'(x_{A_\lambda^{-1}B_\lambda A_\lambda T_{\lambda-1}} - x_{T_{\lambda-1}}) + \sum_{\lambda=1}^{g} {}^t\tilde{y}_{B_\lambda} l'(x_{B_\lambda A_\lambda T_{\lambda-1}} - x_{A_\lambda^{-1}B_\lambda A_\lambda T_{\lambda-1}}) + \sum_{\mu=1}^{j} {}^t\tilde{y}_{C_\mu^{-1}} l'\Bigg[e_\mu^{-1}\sum_{m=1}^{e_\mu-1} x_{C_\mu^m} - x_{C_{\mu-1}\cdots C_1 T_g}\Bigg] - \sum_{\nu=1}^{k} {}^t\tilde{y}_{D_\nu^{-1}} l'[x_{D_{\nu-1}\cdots D_1 C_j\cdots C_1 T_g}] + \sum_{\nu=1}^{k} {}^t\tilde{y}_{D_\nu^{-1}} l'(\mathrm{Re}\,\mathfrak{f}(s_\nu))\Bigg] > 0.$$

Proof. Set

$$\Phi_1 = \sum_{\lambda=1}^{g} {}^t\tilde{x}_{A_\lambda^{-1}} l'[x_{A_\lambda^{-1}B_\lambda A_\lambda T_{\lambda-1}} - x_{T_{\lambda-1}}] + \sum_{\lambda=1}^{g} {}^t\tilde{x}_{B_\lambda} l'[x_{B_\lambda A_\lambda T_{\lambda-1}} - x_{A_\lambda^{-1}B_\lambda A_\lambda T_{\lambda-1}}] + \sum_{\mu=1}^{j} {}^t\tilde{x}_{C_\mu^{-1}} l'\Bigg[e_\mu^{-1}\sum_{m=1}^{e_\mu-1} x_{C_\mu^m} - x_{C_{\mu-1}\cdots C_1 T_g}\Bigg] - \sum_{\nu=1}^{k} {}^t\tilde{x}_{D_\nu^{-1}} l'[x_{D_{\nu-1}\cdots D_1 C_j\cdots C_1 T_g}] + \sum_{\nu=1}^{k} {}^t\tilde{x}_{D_\nu^{-1}} l'\,\mathrm{Re}\,\mathfrak{f}(s_\nu),$$

$$\Phi_2 = \sum_{\lambda=1}^{g} {}^t\tilde{x}_{A_\lambda^{-1}} l'[y_{A_\lambda^{-1}B_\lambda A_\lambda T_{\lambda-1}} - y_{T_{\lambda-1}}] + \sum_{\lambda=1}^{g} {}^t\tilde{x}_{B_\lambda} l'[y_{B_\lambda A_\lambda T_{\lambda-1}} - y_{A_\lambda^{-1}B_\lambda A_\lambda T_{\lambda-1}}] + \sum_{\mu=1}^{j} {}^t\tilde{x}_{C_\mu^{-1}} l'\Bigg[e_\mu^{-1}\sum_{m=1}^{e_\mu-1} y_{C_\mu^m} - y_{C_{\mu-1}\cdots C_1 T_g}\Bigg] - \sum_{\nu=1}^{k} {}^t\tilde{x}_{D_\nu^{-1}} l'[y_{D_{\nu-1}\cdots D_1 C_j\cdots C_1 T_g}] + \sum_{\nu=1}^{k} {}^t x_{D_\nu^{-1}} l'\,\mathrm{Im}\,\mathfrak{f}(s_\nu),$$

$$\Phi_3 = \sum_{\lambda=1}^{g} {}^t\tilde{y}_{A_\lambda^{-1}} l'[x_{A_\lambda^{-1}B_\lambda A_\lambda T_{\lambda-1}} - x_{T_{\lambda-1}}] + \sum_{\lambda=1}^{g} {}^t\tilde{y}_{B_\lambda} l'[x_{B_\lambda A_\lambda T_{\lambda-1}} - x_{A_\lambda^{-1}B_\lambda A_\lambda T_{\lambda-1}}] + \sum_{\mu=1}^{j} {}^t\tilde{y}_{C_\mu^{-1}} l'\Bigg[e_\mu^{-1}\sum_{m=1}^{e_\mu-1} x_{C_\mu^m} - x_{C_{\mu-1}\cdots C_1 T_g}\Bigg] - \sum_{\nu=1}^{k} {}^t\tilde{y}_{D_\nu^{-1}} l'[x_{D_{\nu-1}\cdots D_1 C_j\cdots C_1 T_g}] + \sum_{\nu=1}^{k} {}^t\tilde{y}_{D_\nu^{-1}} l'\,\mathrm{Re}\,\mathfrak{f}(s_\nu),$$

$$\Phi_4 = \sum_{\lambda=1}^{g} {}^t\tilde{y}_{A_\lambda^{-1}} l'[y_{A_\lambda^{-1}B_\lambda A_\lambda T_{\lambda-1}} - y_{T_{\lambda-1}}] + \sum_{\lambda=1}^{g} {}^t\tilde{y}_{B_\lambda} l'[y_{B_\lambda A_\lambda T_{\lambda-1}} - y_{A_\lambda^{-1}B_\lambda A_\lambda T_{\lambda-1}}]$$
$$+ \sum_{\mu=1}^{j} {}^t\tilde{y}_{C_\mu^{-1}} l'\left[e_\mu^{-1}\sum_{m=1}^{e_\mu-1} y_{C_\mu^m} - y_{C_{\mu-1}\cdots C_1 T_g}\right] - \sum_{\nu=1}^{k} {}^t\tilde{y}_{D_\nu^{-1}} l'[y_{D_{\nu-1}\cdots D_1 C_j \cdots C_1 T_g}]$$
$$+ \sum_{\nu=1}^{k} {}^t\tilde{y}_{D_\nu^{-1}} l' \operatorname{Im} \mathfrak{f}(s_\nu).$$

Combining the equations (II) and (III) in the case of $\phi = \psi \neq 0$, we have

$$\Phi_1 + i\Phi_2 - i\Phi_3 + \Phi_4 = 2i(-1)^{q-1}\|\phi\|^2$$

and

$$\Phi_1 + i\Phi_2 + i\Phi_3 - \Phi_4 = 0.$$

Thus $\Phi_1 = \Phi_4 = 0$ and $\Phi_3 = -\Phi_2 = (-1)^q\|\phi\|^2$. Since $(-1)^q\Phi_3 > 0$, we have the desired result.

Remark. Especially, when $q = 1$,

$$(-1)\sum_{\lambda=1}^{g}(y_{B_\lambda}x_{A_\lambda} - y_{A_\lambda}x_{B_\lambda}) > 0.$$

This is the period inequality for abelian integrals.

The following result of Kra [5] is obtained from the above corollary.

Corollary 3. *Let Γ be the same group as in the above corollary. If X_A is real for every $A \in \Gamma$, then $X_A = 0$.*

Proof. In Corollary 2 to Theorem 2, we have $y_{A_\lambda} = 0$, $y_{B_\lambda} = 0$ $(\lambda = 1, \cdots, g)$, $y_{C_\mu} = 0$ $(\mu = 1, \cdots, j)$ and $y_{D_\nu} = 0$ $(\nu = 1, \cdots, k)$, and so ϕ must be zero. Hence $X_A = 0$.

Finally we consider meromorphic Eichler integrals. We denote by $M_{1-q}(\Delta, \Gamma)$ the space of identified meromorphic Eichler integrals. Then we have the following:

Theorem 3. *Let Γ be the same group as in Theorem 1. Assume, for the sake of simplicity, that the group has neither parabolic nor elliptic elements. Let $f \in M_{1-q}(\Delta, \Gamma)$, $q \geq 1$, and E an arbitrary representative of f. Let E have only one pole at u_1 in ω_0 with principal part $(1/z^m)$, $m \geq 1$. Let E^* be an arbitrary representative of f^*, $f^* \in E_{1-q}(\Delta, \Gamma)$ such that $D^{2q-1}E^* = \phi^* \in B_q(\Delta, \Gamma)$ has the representation $\phi^*(z) = (c_0 + c_1 z + \cdots)dz^q$ about u_1. Let $\mathrm{pd}_A\, \mathfrak{f} = X_A$ and*

$\mathrm{pd}_A\ \mathfrak{f}^* = X_A^*$ *for each* $A \in \Gamma$, *where* $\mathfrak{f}$ *and* $\mathfrak{f}^*$ *are column vectors of length* $2q - 1$ *of the form* (1) *associated with* E *and* E^*, *respectively. Then*

$$\sum_{\lambda=1}^{g} [({}^t\widetilde{X}_{A_\lambda^{-1}} I' X_{B_\lambda}^* - {}^t\widetilde{X}_{B_\lambda^{-1}} I' X_{A_\lambda}^*) + {}^t(\widetilde{X}_{A_\lambda} - \widetilde{X}_{B_\lambda^{-1}}) I' M(A_\lambda) X_{T_{\lambda-1}}^*$$
$$\qquad {}^t(\widetilde{X}_{A_\lambda^{-1}} - \widetilde{X}_{B_\lambda}) I' M(B_\lambda) X_{T_\lambda}^*] - 2\pi i n! c_{m-1}.$$

Proof. We easily see that

$$\int {}^t\mathfrak{f}(z) I' \left(\frac{1}{z}\right)^n \phi^*(z)\, dz = n! \int E(z) \phi^*(z)\, dz = 2\pi i n! c_{m-1}.$$

By using the same way as in Theorem 2, we see that the left-side hand is equal to

$$\sum_{\lambda=1}^{g} [({}^t\widetilde{X}_{A_\lambda^{-1}} I' X_{B_\lambda}^* - {}^t\widetilde{X}_{B_\lambda^{-1}} I' X_{A_\lambda}^*) + ({}^t(\widetilde{X}_{A_\lambda} - \widetilde{X}_{B_\lambda^{-1}}) I' M(A_\lambda) X_{T_{\lambda-1}}^*)$$
$$\qquad + {}^t(\widetilde{X}_{A_\lambda^{-1}} - \widetilde{X}_{B_\lambda}) I' M(B_\lambda) X_{T_\lambda}^*].$$

Remark. Let $\widetilde{\Delta}$ be a finite sum of nonequivalent simply connected components $\Delta_1, \Delta_2, \cdots, \Delta_t$, that is, $\widetilde{\Delta} = \bigcup_{i=1}^{t} \Delta_i$, where $\Delta_i \neq A(\Delta_j)$ $(i \neq j)$ for any $A \in \Gamma$. Then we obtain similar results as above.

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DEPARTMENT OF MATHEMATICS, SHIZUOKA UNIVERSITY, OHYA SHIZUOKA, JAPAN